7. G. D. Rabinovich, R. Ya. Gurevich, and G. I. Bobrova, Separation of Liquid Mixtures by Thermal Diffusion [in Russian], Minsk (1971).
8. C. Jones and W. Ferry, Isotope Separation by Thermal Diffusion [Russian translation], Moscow (1947).
9. V. A. Nelidov, Doctoral Dissertation, Leningrad (1973).
10. C. N. Satterfield and T. K. Sherwood, The Role of Diffusion in Catalysis, Addison-Wesley (1963).
11. S. Dicave and A. Emery, AIChE J., 13, No. 6 (1967).
12. V. Sanchez, J. Mahenc, and M. Sagnes, Chim. Ind. Genie Chimique, 103, 2407 (1970).
13. J. Estebe, Chim. Ind. Genie Chimique, 104, No. 3, 355 (1971).
14. H. Z. Korsching, Naturforsch., 109, 242 (1955).
15. M. J. Story and J. C. Turner, Trans. Faraday Soc., 65, 349 (1969).
16. L. Guczi and J. V. Tyrrell, J. Chem. Soc., 6576-6586 (1965).
17. L. J. Tichacek, W. S. Kmak, and H. G. Drickamer, J. Phys. Chem., 60, No. 5, 660-665 (1956).
18. E. von Halle, AEC Research and Development Report K-1420 (1959).
19. Landolt–Börnstein, Vol. 5, Part a, Berlin (1969).

# SOLUTION OF BOUNDARY-VALUE PROBLEMS OF HEAT CONDUCTION FOR CYLINDRICAL REGIONS WITH NONCIRCULAR BOUNDARIES

D. Ya. Byalik and Yu. I. Solov'ev

UDC 536.24

We employ the principle of superposition to obtain the solution of stationary heat-conduction problems for cylindrical regions with a noncircular boundary.

We consider the problem of the stationary temperature distribution in an infinite cylinder with a noncircular contour

$$\rho = f(\theta), \tag{1}$$

where $\rho$, $\theta$, and z are cylindrical coordinates (the z axis coincides with the axis of the cylinder). We assume the cylinder contour to be convex and smooth [the derivative $f'(\theta)$ is continuous]. The known surface temperature is constant along the contour; along a generator of the cylinder it varies as cos nz.

Thus, we solve the following three-dimensional boundary-value problem of the theory of heat conduction: Find a function $u(\rho, \theta, z)$ satisfying Laplace's equation

$$\Delta u = 0 \tag{2}$$

and the boundary condition

$$u|_{\rho=f(\theta)} = \cos nz \quad (n = 1, 2, \ldots). \tag{3}$$

To obtain such a solution we use the principle of superposition (see [1]). In the space of the coordinates x, y, and z we select a new system of coordinates X, Y, and z, which depends on the parameter $\lambda$ and is defined by the expressions

$$\begin{cases} X = x\cos\lambda + y\sin\lambda, \\ Y = -x\sin\lambda + y\cos\lambda, \\ z = z. \end{cases} \tag{4}$$

Novosibirsk Cooperative Trade Institute. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 30, No. 5, pp. 914-917, May, 1976. Original article submitted December 20, 1974.

In this variable coordinate system we obtain a solution of the following two-dimensional boundary-value problem: find a function $v(X, z)$, satisfying the equation

$$\left(\frac{\partial^2}{\partial X^2}+\frac{\partial^2}{\partial z^2}\right)v=0 \tag{5}$$

in the infinite strip $X = \pm a$, subject to boundary conditions analogous to the conditions (3), namely,

$$v_{X=\pm a}=\cos nz. \tag{6}$$

Using the Fourier method, we readily find that

$$v(X, z)=\frac{\operatorname{ch} nX}{\operatorname{ch} na}\cos nz, \tag{7}$$

and, taking Eqs. (4) into account, we then have

$$v(x, y, \lambda)=\frac{\operatorname{ch} n(x\cos\lambda+y\sin\lambda)}{\operatorname{ch} na}\cos nz. \tag{8}$$

For every value of the parameter $\lambda$ this function constitutes a solution of the three-dimensional Laplace equation (2) in Cartesian coordinates. Using the superposition principle (see [1]), we seek the solution of the problem (2), (3) in the form of an integral, with respect to the parameter $\lambda$, of the solution (7) of the two-dimensional problem:

$$u(x, y, z)=\cos nz\int_0^{2\pi}\psi(\lambda)v(x, y, \lambda)\,d\lambda \tag{9}$$

or

$$u(x, y, z)=\frac{\cos nz}{\operatorname{ch} na}\int_0^{2\pi}\psi(\lambda)\operatorname{ch} n(x\cos\lambda+y\sin\lambda)\,d\lambda. \tag{10}$$

It is readily seen that the expression (10) satisfies Eq. (2). In order to satisfy the boundary condition (3) we rewrite Eq. (10) in cylindrical coordinates:

$$u(\rho, \theta, z)=\cos nz\int_0^{2\pi}\psi(\lambda)\operatorname{ch}[n\rho\cos(\lambda-\theta)]\,d\lambda \tag{11}$$

[the unknown function $\psi(\lambda)$ in this expression is $\cosh na$ times less than the $\psi(\lambda)$ in Eq. (10)]. Substituting Eq. (11) into Eq. (3), we obtain a Fredholm integral equation of the first kind in the unknown function; thus,

$$\int_0^{2\pi}\operatorname{ch}[nf(\theta)\cos(\lambda-\theta)]\,\psi(\lambda)\,d\lambda=1. \tag{12}$$

We employ numerical methods to find the function $\psi(\lambda)$ [2]. We put

$$K(\theta, \lambda)=\operatorname{ch}[nf(\theta)\cos(\lambda-\theta)]=\sum_{r,s=0}^{\infty}a_{rs}\cos r\theta\cos s\lambda,$$
$$1=\sum_{r=0}^{\infty}b_r\cos r\theta,\quad \psi(\lambda)=\sum_{r=0}^{\infty}c_r\cos r\lambda \tag{13}$$

and obtain the unknown coefficients

$$a_{rs}=\frac{1}{\pi^2}\int_0^{2\pi}\cos r\theta d\theta\int_0^{2\pi}\operatorname{ch}[nf(\theta)\cos(\lambda-\theta)]\cos s\lambda d\lambda$$
$$=\frac{2}{\pi}\int_0^{2\pi}\cos r\theta(-1)^s\cos\left(\frac{\pi s}{2}\right)J_s(nif(\theta))\cos s\theta d\lambda=\frac{2}{\pi}(-1)^s\cos\left(\frac{\pi s}{2}\right)\int_0^{2\pi}J_s(nif(\theta))\cos r\theta\cos s\theta d\theta.$$

Here the factor $\cos(\pi s/2)$ is equal to zero when s is odd. Taking this fact into account, and taking note also of the relationship between the Bessel functions of real and imaginary arguments, namely, $J_s(iz) = i^s I_s(z)$ (see [3]), we obtain

$$a_{rs} = \frac{2}{\pi} \int_0^{2\pi} I_s(nf(\theta)) \cos r\theta \cos s\theta d\theta$$

or

$$a_{rs} = \frac{n^s}{\pi 2^{s-1}} \sum_{k=0}^{\infty} \frac{n^{2k} 2^{-2k}}{k!\,(k+s)!} \int_0^{2\pi} [f(\theta)]^{2k+s} \cos r\theta \cos n\theta d\theta, \tag{14}$$

$$r = 0,\ 1,\ 2,\ \ldots,\ s = 0,\ 2,\ 4,\ \ldots.$$

As before, $f(\theta)$ is the equation of the contour of the cylinder in cylindrical coordinates. Further, we have

$$b_0 = 1,\ b_r = 0,\ r = 1,\ 2,\ \ldots. \tag{15}$$

Substituting Eqs. (13) into Eq. (12), we reduce the integral equation to an infinite system of linear equations in the unknowns $c_s$; for computational purposes we replace this infinite system by the finite system

$$\sum_{s=0,\,2,\,4,\,\ldots}^{2m} a_{rs} c_s = b_r,\ r = 0,\ 1,\ 2,\ \ldots,\ m. \tag{16}$$

The coefficients $c_s$ obtained from the system (16) determine the unknown function $\psi(\lambda)$, which determines, in turn, through the expression (11), the temperature field inside the cylinder.

In the special case of a cylinder with a circular contour $\rho = R$, the method presented above leads to a known solution in closed form. We have $f(\theta) = R$. We put $\psi(\lambda) = A$ and evaluate the integral on the left side of Eq. (12):

$$A \int_0^{2\pi} \operatorname{ch}[nR\cos(\lambda - \theta)]\, d\lambda,$$

which, after the substitutions $\lambda - \theta = t$, $\cos t = \zeta$, becomes

$$-4A \int_0^1 \frac{\operatorname{ch}(nR\zeta)\, d\zeta}{\sqrt{1-\zeta^2}} = -2A\pi I_0(nR).$$

From this we find

$$A = \frac{-1}{2\pi I_0(nR)}.$$

Substituting this value into Eq. (11), and again evaluating the same type of integral, we obtain

$$u(\rho,\ \theta,\ z) = \frac{I_0(n\rho)}{I_0(nR)} \cos nz,$$

which is the same as the solution obtained by the method of separating the variables.

## NOTATION

$\rho$, $\theta$, $z$, cylindrical coordinates; $u(\rho, \theta, z)$, temperature at point $(\rho, \theta, z)$; $v(X, z)$, infinite plate temperature; $\lambda$, continuous parameter; $J_s(nif(\theta))$, Bessel function of the s-th order; $i = \sqrt{-1}$; $\rho = f(\theta)$, equation for a cylinder contour; $I_s(z)$, Bessel function of the s-th order of imaginary argument; $R$, radius of a circular cylinder.

## LITERATURE CITED


1. P. Frank and R. von Mises, Differential and Integral Equations of Mathematical Physics [Russian translation], Moscow (1937).
2. K. S. Kunz, Numerical Analysis, New York (1957).
3. E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Cambridge University Press, Cambridge (1920).